CALCULUS (FDC 112C)

Course Objectives

By the end of the course, students will be able to:

- ❖ Explain the concept ‘limit’ of a function; and use this understanding to find the limits of a function involving infinity, and continuous functions;
- ❖ Use the idea of limit to find the derivatives of a function from first principle and its applications to polynomials, and gradients of a curve;
- ❖ Differentiate functions of sums, of products, and of quotients;
- ❖ Apply derivatives to determine maxima and minima so as to sketch a curve; to determine the gradient and hence equation of tangent and normal to a curve; apply the knowledge of derivatives to finding distance, velocity and acceleration;
- ❖ Integrate functions, including definite and indefinite integrals;
- ❖ Integrate function of functions;
- ❖ Apply integration to finding the area under a curve, and area bounded by two curves; and application to distance, velocity and acceleration.

UNIT 1

The concept of Limit

The limit theorem describes what happens to a function, $f(x)$, as x approaches a particular number, called ‘c’, within its domain.

Illustration: What happens to the function, $f(x) = \dfrac{x^2 + x - 2}{x - 1}$ as x approaches 1, written as: $\lim\limits_{x \to 1} \frac{x^2+x-2}{x-1}$, $x \neq 1$.

Note: that the function, $f(x)$ is not defined at $x = 1$; so, we consider values nearer and very close to 1, that is, before getting to 1, and immediately after 1. Let us investigate the behaviour of the function, $f(x)$ as x approaches 1. This is summarized in the table below.

x	0.9	0.95	0.99	0.999	1	1.001	1.01	1.05	1.10
$f(x)$	2.9	2.95	2.99	2.999	∞	3.001	3.01	3.05	3.10

From the table, it can be seen that as x approaches 1 from either side, $f(x)$ approaches 3 but notice what happens at 1*!!!* At 1, the function is NOT defined, we say, the function is ***UNDEFINED***.

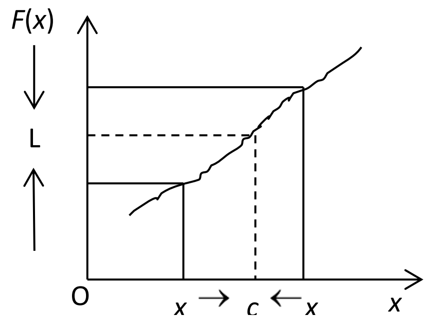
The idea or concept of limit is to ask the simple question, assuming x is very very close getting to 1, what will be the greatest possible value the function will take? That value, as we can see from the table, is 3. Hence, the limit of the function 3, denoted as $\lim\limits_{x \to 1} f(x) = 3$

Limit Theorem (Let us look at the theoretical approach).

If x gets closer and closer to a number, 'c', from either side, $f(x)$ gets closer and closer to a particular number 'L', therefore, L becomes the limit of the function, $f(x)$. i.e., $\lim_{x\to c} f(x) = L$.

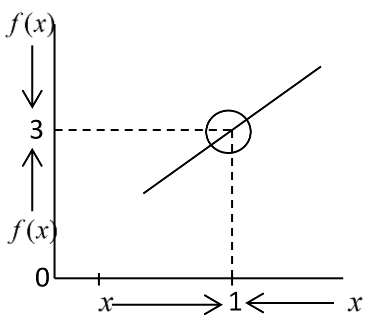
Geometrically it talks about the height of the graph of $y = f(x) \to L$; $y = L$ as $x \to c$.

This is illustrated below:



Example: $\lim_{x\to 1} f(x) = \frac{x^2+x-2}{x-1} = 3 \ldots$ [***We will revisit the techniques of finding limit***].

This is illustrated in the diagram below:



From the diagram, it shows clearly that as x approaches 1, the function, *f(x)*, also approaches 3. i.e., $\lim_{x\to 1} f(x) = 3$

So, what have you learned about limits? Take notice that, in general, limits are asking the question, what is the function doing **AROUND** $x = c$ and **NOT** concerned with what the function **is actually doing AT** $x = c$. This is a good thing as many of the functions that we will be looking at would not even exist at $x = c$.

Summary of the Properties of Limit

Sum, Difference, Multiple or Product.

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then:

1. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c}[kf(x)] = k\lim_{x\to c} f(x)$ for any constant k.
4. $\lim_{x\to c}[f(x)\bullet g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)]$.

Limit of Quotient

5. If the $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do exist, then:

$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \left[\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}\right]$, where $g(x) \neq 0$, i.e., if the denominator is not zero.

Limit of Power

6. If $\lim_{x\to c} f(x) = L$ and p is a real number for which L is defined then

$\lim_{x\to c}[f(x)]^p = [\lim_{x\to c} f(x)]^p = L^p$

Limits of Polynomials and Rational Functions.

7. If $p(x)$ and $q(x)$ are polynomials, then $\lim_{x\to c} p(x) = p(c)$ and $\lim_{x\to c}\frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$;

for which $q(x) \neq 0$.

Limits of function of function

8. If $\lim_{x\to c} g(x) = B$, then $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(B)$; provided $f(g(x))$ is continuous at c.

Worked Examples

1. Find the $\lim_{x\to 2}\left(\frac{x+1}{x-3}\right)$ ……….. [A **Simple Case**]

Solution: To find the limit, substitute the value of *x* into the expression as below:

$\lim_{x\to 2}(x-3) = -1$ = -1, and $\lim_{x\to 2}(x+1) = 3$

$\therefore \lim_{x\to 2}\left(\frac{x+1}{x-3}\right) = \frac{3}{-1} = -3$

2. Find the $\lim_{x\to 1}\left(\frac{3x-1}{x-1}\right)$

Solution

$\lim_{x\to 1}(3x-1) = 2$, and $\lim_{x\to 1}(x-1) = 0$

Since the $\lim_{x\to 1}\left(\frac{3x-1}{x-1}\right) = \frac{2}{0} = -\infty$, $\therefore$ the limit is infinity.

Use your calculator to investigate this, by taking values of *x* approaching 1, and observe the values the function.

3. Find the $\lim_{x\to 1}\left(\dfrac{x^2-1}{x^2-3x+2}\right)$ ….. [**A Hard Case**]

Solution

Here both the numerator and the denominator approach zero so we factorize each of them to get:

$$\lim_{x\to 1}\left(\frac{x^2-1}{x^2-3x+2}\right)=\lim_{x\to 1}\left[\frac{(x+1)(x-1)}{(x-2)(x-1)}\right]$$

$$= \lim_{x\to 1}\left(\frac{x+1}{x-2}\right)$$

$$\lim_{x\to 1}\left(\frac{x^2-1}{x^2-3x+2}\right)=\frac{\lim_{x\to 1}(x+1)}{\lim_{x\to 1}(x-2)}=\frac{2}{-1}=-2$$

4. Find $\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)$ ……. [**A Harder Case**]

Solution

Multiply both the numerator and the denominator by the conjugate of $\sqrt{x}-1$,

i.e. $\sqrt{x}+1$.

$$\Rightarrow \lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)}=\frac{(x-1)}{(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1}\left(\frac{1}{\sqrt{x+1}}\right)=\frac{1}{2}$$

$$\therefore \lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)=\lim_{x\to 1}\left(\frac{1}{\sqrt{x}+1}\right)=\frac{1}{2}$$

Note: Limit of any two linear function. For any number ‘c’ and a constant *k*, $\lim_{x\to c}(k)=k$;

and $\lim_{x\to c}(x)=c$

$\therefore$ Limit of a constant is the constant itself and limit of $f(x)=x$ is c as $x\to c$.

5. Find $\lim_{x\to\sqrt{5}}\left[\dfrac{x^2-5}{x-\sqrt{5}}\right]$

Solution

Rationalize the function, i.e. $\lim_{x\to\sqrt{5}}\left[\dfrac{(x^2-5)(x+\sqrt{5})}{x^2-5}\right]=\lim_{x\to\sqrt{5}}(x+\sqrt{5})$

$$\therefore \lim_{x\to\sqrt{5}}\left(\frac{x^2-5}{x-\sqrt{5}}\right)=\lim_{x\to\sqrt{5}}(x+\sqrt{5})=2\sqrt{5}$$

6. Evaluate the following limits if it exists:

(a) $\lim\limits_{h\to 0}\frac{(3+h)^2-9}{h}$ (b) $\lim\limits_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$ (c). $\lim\limits_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$ (d) $\lim\limits_{x\to 2}\frac{2x^2+1}{x^2+6x-4}$

(e) $\lim\limits_{x\to 2}\frac{x^2+x-6}{x-2}$ (f). $\lim\limits_{x\to -1}\frac{x^2-4x}{x^2-3x-4}$ (g). $\lim\limits_{x\to -2}(3x^4+2x^2-x+1)$

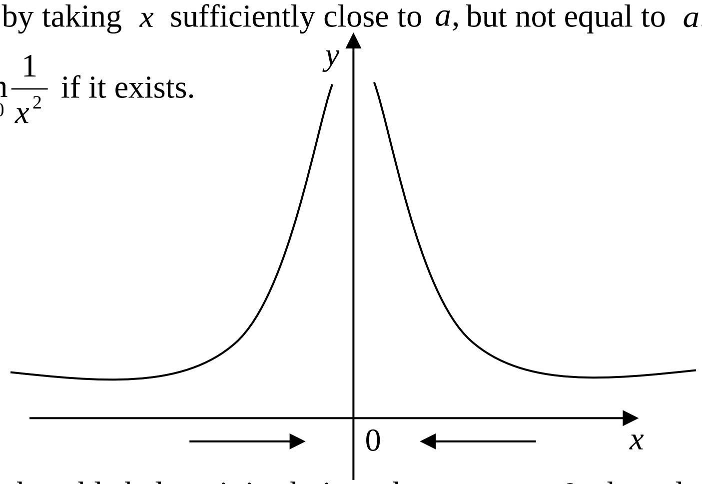
Infinite Limits (Vertical Asymptote)

Definition: Let f be a function defined in the domain of real numbers a, except possibly at a itself. Then:

$$\lim_{x\to a} f(x) = +\infty \qquad \text{or} \qquad \lim_{x\to a} f(x) = -\infty$$

This means that the values of $f(x)$ can be made arbitrarily large or increase without bound (as large as it is possible) by taking x sufficiently close to a, but not equal to a.

Example 1: Find $\lim\limits_{x\to 0}\frac{1}{x^2}$ if it exists.



From the diagram and the table below, it is obvious that, as $x \to 0$, the value of the function approaches infinity.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10000
± 0.001	1000000

Hence, $\lim\limits_{x\to 0}\frac{1}{x^2} = \infty$

Conversely, $\lim\limits_{x\to 0} -\frac{1}{x^2} = -\infty$

Definition: If $\lim_{x \to a} f(x) = \pm\infty$, then the line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$, if at least one of the following statements is true:

(i) $\lim_{x \to a} f(x) = \infty$ (ii) $\lim_{x \to a^-} f(x) = \infty$

(iii) $\lim_{x \to a^+} f(x) = \infty$ (iv) $\lim_{x \to a} f(x) = -\infty$

(v) $\lim_{x \to a^-} f(x) = -\infty$ (vi) $\lim_{x \to a^+} f(x) = -\infty$

Example 2: Find (i): $\lim_{x \to 3^+} \frac{2x}{x-3}$ ……. [i.e., as x approaches 3 from the right]

(ii): $\lim_{x \to 3^-} \frac{2x}{x-3}$ …… [i.e., as x approaches 3 from the left]

Solution: (i): If x is close to 3 but larger than 3, then the denominator $x-3$ is a small positive number (Let us demonstrate this: e.g., if x is 3.0001, then, we have, 3.0001 – 3 = 0.0001) and $2x$ is close to 6 so, the quotient $\frac{2x}{x-3}$ is a large positive number. Thus, intuitively, we see that

$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$ ……. [Intuitively]

Conversely, if x is close to 3 but smaller than 3, then the denominator $x-3$ is a small negative number (Let us demonstrate this: e.g., if x is 2.9999, then, we have, 2.9999 – 3 = – 0.0001) but $2x$ is still a positive number (close to 6). So $\frac{2x}{x-3}$ is a numerically large negative number. Hence,

$\lim_{x \to 3^{--}} \frac{2x}{x-3} = -\infty$ …… [Intuitively]

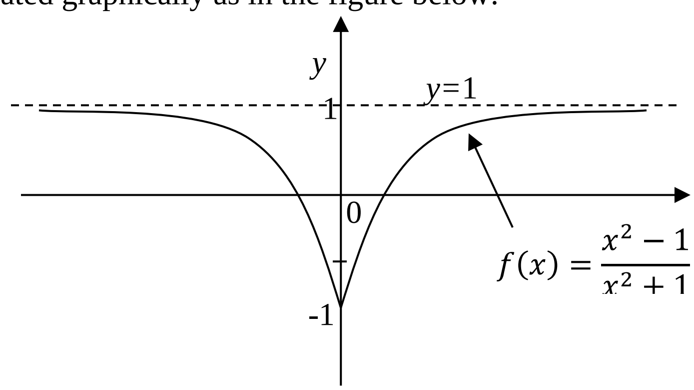
Limit at Infinity (Horizontal Asymptotes)

Unlike the case of vertical asymptotes where we x approaches a number such that the result of $y = f(x)$ becomes arbitrarily large from both sides (positive or negative), in horizontal asymptote, x is considered to take on arbitrarily large (positive or negative) value. In this case, the result of $y = f(x)$ approaches a particular number, as denoted below: $\lim_{x \to \infty} f(x) = L$, For example, if $f(x)=\dfrac{x^2-1}{x^2+1}$

x	$f(x)$
0	- 1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

From the table it can be noticed that as x become very large, $f(x)$ approaches a particular number, 1. This implies that $\lim\limits_{x\to\infty} f(x)=\dfrac{x^2-1}{x^2+1}=1$.

This is demonstrated graphically as in the figure below:



Example: (1) $\lim\limits_{x\to\infty}\left(\dfrac{3x+2}{7x-1}\right)$

Solution

For limit to be infinite then

$$\lim_{x\to\infty}\left(\frac{3x+2}{7x-1}\right)=\lim_{x\to\infty}\left(\frac{\frac{3x}{x}+\frac{2}{x}}{\frac{7x}{x}-\frac{1}{x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{3+\frac{2}{x}}{7-\frac{1}{x}}\right)$$

Note: as $x \to \infty$, then $\frac{2}{x} \to 0$ and $\frac{-1}{x} \to 0$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{3x+2}{7x-1}\right) = \lim_{x\to\infty}\left(\frac{3+\frac{2}{x}}{7-\frac{1}{x}}\right) = \frac{3}{7}$$

2. Evaluate $\lim_{x\to\infty}\left(\frac{3x^2+6}{4x^2-1}\right)$

Solution

$$\lim_{x\to\infty}\left(\frac{3x^2+6}{4x^2-1}\right) = \lim_{x\to\infty}\left(\frac{\frac{3x^2}{x^2}+\frac{6}{x^2}}{\frac{4x^2}{x^2}-\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{3+\frac{6}{x^2}}{4-\frac{1}{x^2}}\right) = \frac{3}{4}$$

Since as $x \to \infty$; $\frac{6}{x^2} \to 0$; and $\frac{-1}{x^2} \to 0$

3. Evaluate (a) $\lim_{x\to\infty}\left(\frac{3x}{x+2}\right)$

(b) $\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right)$

Solution

(a)

$$\lim_{x\to\infty}\left(\frac{3x}{x+2}\right) = \lim_{x\to\infty}\left(\frac{\frac{3x}{x}}{\frac{x}{x}+\frac{2}{x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{3}{1+\frac{2}{x}}\right) = \frac{3}{1} = 3$$

$$\therefore \lim_{x\to\infty}\left(\frac{3x}{x+2}\right) = 3$$

(b) $\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right) =$

$$\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right) = \lim_{x\to 1}\left[\frac{(x-1)(x^2+x+1)}{x-1}\right]$$

$$= \lim_{x\to 1}(x^2+x+1)$$

$$= (1+1+1)$$

$$\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right) = 3$$

4. Find the $\lim\limits_{x\to\infty} g(x)$ given that $g(x) = \dfrac{3x-1}{3x-2} = A + \dfrac{B}{x-2}$ ……… [You should know how to do this by now]

Solution

$\Rightarrow \; 3x-1 = A(x-2)+B$ where $x = 2$

$6-1 = B \quad \Rightarrow \quad B = 5$ for $x = 0$

-1 = -2A +5 $\quad -1 = -2A + 5$

$-2A = -6$

$$A = \frac{-6}{-2} = 3$$

$$\Rightarrow g(x) = 3 + \frac{5}{x-2}$$

OR

$$g(x) = \frac{3x-1}{x-2}$$

$$= x-2 \,\overline{)\,3x-1}\quad 3$$

$$\frac{-3x-6}{+5}$$

$$\Rightarrow \; g(x) = 3 + \frac{5}{x-2}$$

$$\therefore \lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(3 + \frac{5}{x-2}\right) = 3$$

This is because as x becomes infinitely large, then $\dfrac{5}{x-2} \to 0$

Alternatively,

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{3x-1}{x-2}$$

$$= \lim_{x\to\infty}\left[\frac{\frac{3x}{x} - \frac{1}{x}}{\frac{x}{x} - \frac{2}{x}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{3 - \frac{1}{x}}{1 - \frac{2}{x}}\right] = \frac{3}{1}$$

$$\therefore \lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(\frac{3x-1}{x-2}\right) = 3$$

Exercises: 1

Evaluate each of the following limits.

1. $\lim_{x\to 0}\left(\frac{3x^2-8}{x-2}\right)$

2. $\lim_{x\to 0}(x^2-6x+7)$.

3. $\lim_{x\to 0}\left(\frac{x^2+x-6}{x-2}\right)$

4. $\lim_{x\to 0}\left[\frac{x(x^2-1)}{x^2}\right]$

5. $\lim_{x\to -2}\left(\frac{x^3+8}{x+2}\right)$

6. $\lim_{x\to \infty}\left[\frac{(1+x)(2+x)x}{x^3+x}\right]$

7. $\lim_{x\to \infty}\left(\frac{3x^2-5x-1}{7+2x-4x^2}\right)$

8. $\lim_{x\to -1}\left(\frac{x^2+2x+1}{x^2+3x+2}\right)$

9. $\lim_{x\to 2}\left(\frac{x^2+x-6}{x-2}\right)$

10. $\lim_{x\to 1}\left(\frac{x^2+4x-5}{x^2-1}\right)$

11. $\lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right)$

12. $\lim_{x\to 3}\left(\frac{9-x^2}{4-\sqrt{(x^2+7)}}\right)$

UNIT TWO
DIFFERENTIVE OF A FUNCTION

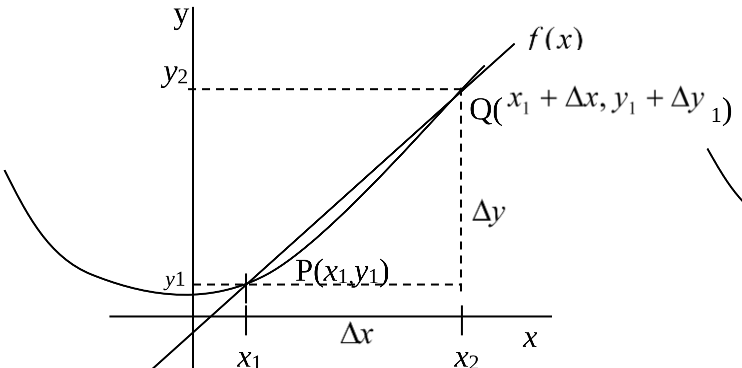
Derivative of a function

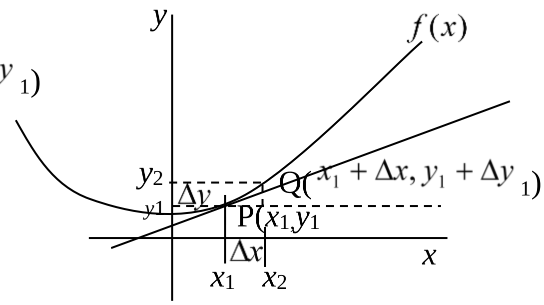
Differentiation is the process of finding the derivative of a function. This is also a mathematics technique of exceptional power and versatility. It is one of the two central concepts in a very important branch of mathematics called calculus which deals with the science of small quantities.

Calculus has a variety of applications such as curve sketching, the optimization of functions and analysis of rate of change, etc.

Definition:

Geometrically, derivative of a function *f* is the derived related function which expresses the slope of the tangent to the curve in terms of the *x* co-ordinate of the point of tangency.





Assuming P (x_1, y_1) is a fixed point on the curve *f(x)*, and Q $(x_1 + \Delta x, y_1 + \Delta y)$ is a variable point on the curve. If the line through P and Q now rotates clockwise about P such that Q gets closer and closer to P to form tangent at P with the condition that both $\Delta x \to 0$, as $x_2 - x_1 \to 0$, and $\Delta y \to O$, as $y_2 - y_1 \to 0$. However, their ratio, which is the gradient of line PQ approaches the gradient of the tangent at P : i.e. $\dfrac{\Delta y}{\Delta x} \to \dfrac{dy}{dx}$.

Therefore, the derivative with respect to x at P is given by

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \qquad = \lim_{\Delta x \to 0}\left[\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}\right] = \frac{dy}{dx}$$

This depicts the first principle which is to be discussed in the next section.

The First Principle

Using the definition above, let $P(x, y)$ be a fixed point on the curve $y = 2x^2 + 6$, and $Q(x + \Delta x, y + \Delta y)$ be a point in the neighbourhood. Notice that Δx and Δy are the respective differences in the x - and y - values of the points P and Q on the curve. The notations dx and dy represent the respective changes in the x - and y -values of two points on a straight line, tangent at P . The quantities dx and dy are called *differentials.*

Example: if *f(x)* = 2*x* + 6, then:

At Q: $y+\Delta y = 2(x+\Delta x)^2+6$

$$= 2(x^2+2x\Delta x+(\Delta x)^2+6$$

$$= 2x^2+4x\Delta x+2(\Delta x)^2+6$$

$$y+\Delta y-y = 2x^2+4x\Delta x+2(\Delta x)^2+6-(2x^2+6)$$

Therefore $\Delta y = 4x\Delta x+2(\Delta x)^2$

Now if we divide through by Δx:

$$\frac{\Delta y}{\Delta x} = 4x+2\Delta x.$$

If the line through P and Q now rotates clockwise about P such that Q gets closer and closer to P to form tangent at P with the condition that both $\Delta x \to O$, and $\Delta y \to O$. However, their ratio, which is the gradient of line PQ approaches the gradient of the tangent at P : $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$.

Thus, $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{\Delta x\to 0}(4x+2\Delta x)$

Therefore, $\frac{dy}{dx} = 4x$.

This is the general result giving the slope of the curve at any point on the curve $y = 2x^2+6$. For example, at the point $x = 1.4$, the gradient

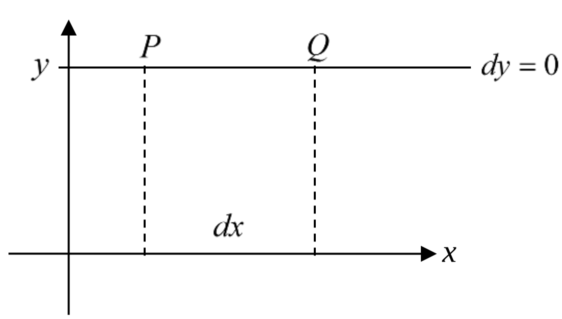
$m = \frac{dy}{dx} = 4(1.4) = 5.6.$

2.2 Derivatives of powers of x

Two straight lines

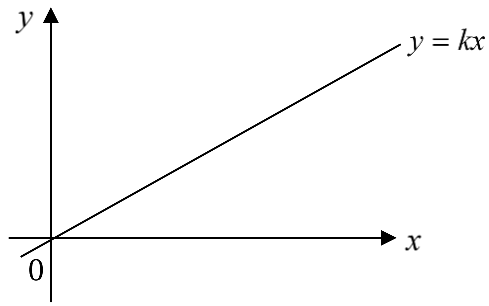
1. $y = c$ (constant)

The graph of this function is a straight line parallel to $x-$ axis with gradient which is equal to zero. Consider the diagram below for further analysis.



$y_2 - y_1 = 0$, this means that $dy = 0$, i.e. a change in the y - values of two points P and Q on line PQ is zero. Therefore, the derivative of the function, $y = c$, $\frac{dy}{dx} = \frac{0}{dx} = 0$

2. Consider the graph of $y = kx$



$$\begin{aligned} y &= kx \\ y + dy &= k(x + dx) \\ &= kx + kdx \\ dy &= kx + kdx - kx \\ &= kdx \\ \therefore \frac{dy}{dx} &= \frac{kdx}{dx} = k \end{aligned}$$

This means that the gradient of a straight line is constant along its length.

If $y = kx$**,** then $\frac{dy}{dx} = k$ **.**

In particular, as in the worked examples 1, if $k = 1$

then $y = x$, **therefore** $\frac{dy}{dx} = 1$ **as in worked example 3 bellow.**

Other worked examples:

From the first principle find the derivative of each of the following functions with respect to x

Example 3.

If $y = x$, find its the derivative.

Solution

If there is small increment in x (independent variable). There would be a correspondent increase in y (dependent variable), but $y = x$.

$\Rightarrow y + \Delta y = x + \Delta x$, where Δx and Δy denotes small increment in *x* and *y*.

$\Rightarrow \Delta y = x + \Delta x - y$

Remember, $y = x$

$\Rightarrow \Delta y = x + \Delta x - x$

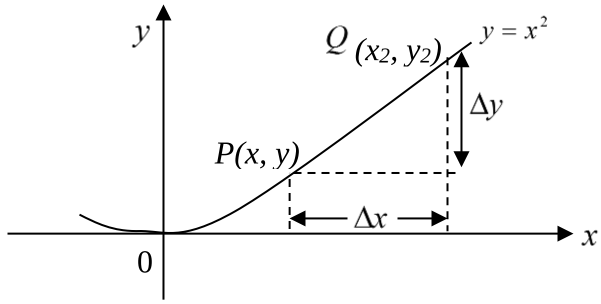
$\Delta y = \Delta x$

$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x}$, (dividing through by Δx)

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (1)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 1$$

4. $y = x^2$



$y + \Delta y = (x + \Delta x)^2$, by expansion, we have:

$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$, making Δy the subject we have;

$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - y$, but $y = x^2$

$$\Rightarrow \Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - x^2$$

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

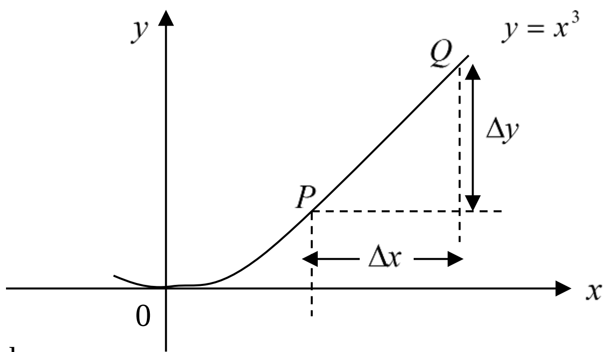
$$\frac{\Delta y}{\Delta x} = \frac{2x\Delta + (\Delta x)^2}{\Delta x}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = 2x + \Delta x$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta \to 0} \left(2x + \Delta x\right)$$

$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x.$ (This is also the gradient of the tangent to the curve).

5. $y = x^3$



From the graph

$$y = x^3$$

$$y + \Delta y = (x+\Delta x)^3$$

$$\Delta y = (x+\Delta x)^3 - y$$

$$\Delta y = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3$$

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$$

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[3x^2 + 3x\Delta x + (\Delta x)^2\right]$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 3x^2$$

6. $y = \dfrac{1}{x}$

$$y + \Delta y = \frac{1}{x+\Delta x} \quad \text{and} \quad \Delta y = \frac{1}{x+\Delta x} - y$$

$$\Delta y = \frac{1}{x+\Delta x} - \frac{1}{x} = \frac{x-(x+\Delta x)}{(x+\Delta x)x}$$

$$\Delta y = \frac{-\Delta x}{(x+\Delta x)x}$$

$$\frac{\Delta y}{\Delta x} = \frac{-\Delta x}{(x+\Delta x)x.} \bullet \frac{1}{\Delta x}$$, hence $$\lim_{\Delta x \to}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{-1}{x+\Delta x)x}\right)$$

$$= \frac{-1}{(x)(x)} = -\frac{1}{x^2}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = -\frac{1}{x^2}$$

7. $y=\frac{1}{x^2}$

$$\text{y + } \Delta y=\frac{1}{(x+\Delta x)^2}$$

$$\Delta y=\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}$$

$$\Delta y=\frac{1}{x^2+2x\Delta x+(\Delta x)^2}-\frac{1}{x^2}$$

$$=\frac{x^2-(x^2+2x\Delta x+(\Delta x)^2.)}{[x^2+2x\Delta x+(\Delta x)^2]x^2}$$

$$\Delta y=\frac{x^2-x^2-2x\Delta x-(\Delta x)^2}{[x^2+2x\Delta x+(\Delta x)^2]x^2}$$

$$\frac{\Delta y}{\Delta x}=\left[\frac{-2x\Delta x-(\Delta x)^2}{\left[x^2+2x\Delta x+(\Delta x)^2\right]x^2}\right]\frac{1}{\Delta x}$$

$$=\frac{-2x-\Delta x}{\left[x^2+2x\Delta x+(\Delta x)^2\right]x^2}$$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left[\frac{-2x-\Delta x}{\left[x^2+2x\Delta x+(\Delta x)^2\right]x^2}\right]$$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left[\frac{-2x-\Delta x}{\left[x^2+2x\Delta x+(\Delta x)^2\right]x^2}\right]$$

$$\therefore\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=\frac{-2x}{x^2(x^2)}=-\frac{2}{x^3}$$

8. $y=3x^2+2$

$$y+\Delta y=3(x+\Delta x)^2+2$$

$$\Delta y=3[x^2+2x\Delta x+(\Delta x)^2+2-y]$$

$$\Delta y=3x^2+6x\Delta x+3(\Delta x)^2+2-(3x^2+2)$$

$$\Delta y=6x\Delta x+3(\Delta x)^2$$

$$\Rightarrow\frac{\Delta y}{\Delta x}=\left[6x\Delta x+3(\Delta x)^2\right]\frac{1}{\Delta x}$$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(6x+3\Delta x)\qquad\qquad\therefore\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=6x$$

9. $y = 1 - 2x$

$$y + \Delta y = 1 - 2(x + \Delta x)$$

$$\Delta y = 1 - 2x - 2\Delta x - y$$

$$\Delta y = 1 - 2x - 2\Delta x - (1 - 2x)$$

$$\Delta y = 1 - 2x - 2\Delta x - 1 + 2x$$

$$\Delta y = -2\Delta x$$

$$\frac{\Delta y}{\Delta x} = \frac{-2\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}(-2)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = -2$$

10. $y = x^2 + 7x - 1$

$$y + \Delta y = (x + \Delta x)^2 + 7(x + \Delta x) - 1$$

$$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 + 7x + 7\Delta x - 1 - x^2$$
$$- 7x + 1$$

$$\Delta y = 2x\Delta x + (\Delta x)^2 + 7\Delta x$$

$$\frac{\Delta y}{\Delta x} = \left[2x\Delta x + (\Delta x)^2 + 7\Delta x\right]\frac{1}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(2x + \Delta x + 7\right)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x + 7$$

11. $y = \dfrac{5}{x^2 - 2}$

$$y + \Delta y = \frac{5}{(x + \Delta x)^2 - 2}$$

$$\Delta y = \frac{5}{x^2 + 2x\Delta x + (\Delta x)^2 - 2} - \frac{5}{x^2 - 2} = \frac{\left(x^2 - 2\right).5 - 5[x^2 + 2x\Delta x + (\Delta x)^2 - 2]}{[x^2 + 2x\Delta x + (\Delta x)^2 - 2][x^2 - 2]}$$

$$\Delta y = \frac{5x^2 - 10 - 5x^2 - 10x\Delta x - 5(\Delta x)^2 + 10}{[x^2 + 2x\Delta x + (\Delta x)^2 - 2][x^2 - 2]}$$

$$\frac{\Delta y}{\Delta x} = \left[\frac{-10x\Delta x - 5(\Delta x)^2}{(x^2 + 2x\Delta x + (\Delta x)^2)(x^2 - 2)}\right]\frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[\frac{-10x - 5\Delta x}{(x^2 + 2x\Delta x + (\Delta x)^2 - 2)(x^2 - 2)}\right]$$

$$= \frac{\Delta y}{\Delta x} = \frac{-10x}{(x^2 - 2)(x^2 - 2)}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{-10x}{(x^2 - 2)^2}$$

12. $f : x \to (x+2)^2$.

Method 1 Let

$$f(x) = y$$

$$\Rightarrow y = (x+2)^2$$

$$y + \Delta y = [(x+\Delta x)+2]^2$$

$$\Delta y = (x+\Delta x)^2 + 4(x+\Delta x) + 4 - (x+2)^2$$

$$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 + 4x + 4\Delta x + 4$$

$$- x^2 - 4x - 4$$

$$\Delta y = 2x\Delta x + (\Delta x)^2 + 4\Delta x$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 4$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}(2x + \Delta x + 4)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x + 4 = 2(x+2)$$

Method 2

$$y = (x+2)^2$$

$$y = x^2 + 4x + 4$$

$$y + \Delta y = (x+\Delta x)^2 + 4(x+\Delta x) + 4$$

$$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 + 4x + 4\Delta x + 4 - x^2 - 4x - 4$$

$$\Delta y = 2x\Delta x + (\Delta x)^2 + 4\Delta x$$

$$\frac{\Delta y}{\Delta y} = 2x + \Delta x + 4$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}(2x + \Delta x + 4)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x + 4$$

Exercises 2

Note:

> **The difference between Grade 'A' and Grade 'E' students is that Grade 'A' students are interested in solving all questions especially those that challenge them under Exercises while Grade 'E' students just scout through questions and answers that are someone's ideas and strategies without solving questions especially those that challenge them under Exercises. You are therefore encouraged to solve the following questions.**

From the first principle find the derivative of the following functions.

1. $y = x^2 + 2x + 3$
2. $f(x) = (x-2)^2$
3. $g(x) = \dfrac{3}{x^2+1}$
4. $k(x) = 1 - 6x - x^2)$
5. $y = 3x^2 - x - 11$
6. $f(x) = x^2 - 5$
7. $f(x) = 2x^2$
8. $f(x) = x^2 + 2x + 1.$
9. $y = x^2 + 2x.$
10. $y = x^2 + x$
11. $f(x) = (x+2)^2$
12. $g(x) = x^3 + x$
13. $h(x) = (2x+3)^2$